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ON THE LINEAR GAME PROBLEM OF CONVERGENCE WITH INCOMPLETE INFORMATION

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S. A. CHIGIR'
(Moscow)

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The game problem of convergence under conditions of incomplete information about phase coordinates is considered. The method used here is similar to that proposed in [1 - 4] which makes it possible to reduce the considered problem to that of conflict control analyzed in a suitable functional space. A construction similar to the retrograde procedure described in [5] is used here as an auxiliary device for deriving a stable system of sets that are necessary for the positional solution of a convergence problem by the extremal aiming rule [2 - 4].

1. Let us consider in an n-dimentional Euclidean space R^n the vector linear differential equation

$$dx / dt = A(t) x - u + v, \quad x \in R^n \tag{1.1}$$

where u and v are control actions exercised by the first and second player, respectively, subject to constraints

$$u \in P(t), \quad v \in Q(t) \tag{1.2}$$

It is assumed that sets $P(t)$ and $Q(t)$ in R^n are nonempty and closed, their dependence on t is measurable (see [6]), and that they satisfy the conditions

$$\| u \| \leqslant \alpha(t), \quad \| v \| \leqslant \beta(t) \tag{1.3}$$

for all $u \in P(t)$ and $v \in Q(t)$. Matrix $A(t)$ and functions $\alpha(t)$ and $\beta(t)$ are assumed summable over any finite segment.

A closed target set defined by variables t, x is selected in the space $R^1 \times R^n$. We restrict the problem to that of convergence at instant ϑ, and assume that set M is convex and lies entirely in the hyperplane $t = \vartheta$. The aim of the first player is to bring the phase vector $x[\vartheta]$ onto set M by the appropriate selection of control $u[t]$. The aim of the second player is to prevent this. We assume, as in [2 - 4], that vector $x \in R^n$ is subject to some nonsingular linear transformation (see [3]) which converts Eq.(1.1) to

$$dx / dt = -u + v \tag{1.4}$$

and simultaneously transforms constraints (1.2) and set M in a known way. We retain previous notation for the transformed sets, and note that all assumptions made about sets $P(t)$, $Q(t)$, and M are satisfied by the transformed sets. Functions $\alpha(t)$ and $\beta(t)$ in formulas (1.3) are replaced by some new summable functions for which we also retain previous notation.

We assume that the first player when formulating his controls, knows at every instant t the convex compact set $G[t] \subset R^n$ which contains state $x[t]$ realized up to that instant, at which he selects his control $u[t] \in P(t)$ on the basis of available information. He may then be faced with any Lebesgue realization of $v[t]$ that satisfies constraints (1.2) and is selected by the second player.

Let us specify the nature of variation of the information sets $G[t]$ with time. We assume, as in [2-4], that whatever the instants t_*, $t^* > t_*$, the set $G[t^*]$ consists only of such phase states to which it is possible to translate the phase vector $x[t]$ at instant t^* in conformity with the law (1.4) from points of set $G[t_*]$ by the exercise of controls $u[t]$ and $v[t]$, $t_* \leqslant t \leqslant t^*$, by the first and second player, respectively, in the course of the game. The specified condition of variation of set $G[t]$ means in essence that whatever the instants t_*, $t^* > t_*$ the first player receives at instant t^* complete information on controls $u[t]$ and $v[t]$, $t_* \leqslant t < t^*$, and that he can use such information only for determining more accurately the information sets, but not for formulating control actions $u[t]$. We also restrict the variation of information sets $G[t]$ by the condition

$$G[t] \in \Gamma(t) \tag{1.5}$$

where $\Gamma(t)$ denotes a certain set of convex compacta in R^n whose properties are defined below.

The problem of control of the phase vector $x[t]$ is, thus, replaced by that of control of the information set $G[t]$ which contains the state $x[t]$ realized at instant t, and the condition that ensures the translation of vector $x[\vartheta]$ onto set M now assumes the form

$$G[\vartheta] \subset M \tag{1.6}$$

2. The mathematical formulation of the considered problem is based in [2-4] on the one-to-one correspondence between convex compacta and their support functions, and subsequent absorption of the input problem in a more general problem of control in a suitable functional space. Hilbert space H formed by scalar positive homogeneous functions $h(l)$ with the square integrable over the unit sphere $\Theta = \{l \in R^n : \|l\| \leqslant 1\}$ and norm $\|h\|_H = \langle h, h \rangle^{1/2}$ defined by the scalar product

$$\langle h_1, h_2 \rangle = \int\limits_{\Theta} h_1(l) h_2(l)\, d\{l\} \tag{2.1}$$

where $d\{l\}$ is the Lebesgue measure in R^n, was taken in [2-4] as such functional space.

The indicated absorption is then obtained in the following manner. We denote by $\mu(l)$ the support function of set M and introduce the set

$$L = \{h \in H : h(l) \leqslant \mu(l), \quad l \in R^n\} \tag{2.2}$$

which in the generalized problem of control in space H represents the target. It is assumed that (2.2) and all subsequent formulas for functions $h(l)$ are satisfied for almost all l. The pairs $\{t, h\}$ are called positions.

Let $\Gamma(t)$ be a one-parametric collection of sets in H that satisfy the following conditions:

$$\begin{aligned} &\Gamma(t^*) \subset \Gamma(t_*), \quad (t^* > t_*) \\ &h_*(l) \leqslant h^*(l), \quad h^*(\cdot) \in \Gamma(t) \Rightarrow h_*(\cdot) \in \Gamma(t) \\ &h_*(l) = h^*(l) + l \cdot x, \quad h^*(\cdot) \in \Gamma(t) \Rightarrow h_*(\cdot) \in \Gamma(t) \end{aligned} \tag{2.3}$$

The first of these defines the accumulation of information about the system in the course of time, and the remaining formalize the properties of information sets. This means that, if at instant t a certain set $G^* \subset R^n$ can be obtained as an information set, then at the same instant any set $G_* \subset G^*$ and, also, any set G_* obtained from G^* by a shift by vector $x \in R^n$ can be realized at the same instant.

Let $g_t(l)$ be some function dependent on parameter t which for any $t \geqslant t_*$ represents an element of space H and satisfies the following conditions:

$$
\begin{aligned}
& g_{t_*}(l) = h_*(l) \\
& g_t(\cdot) \in \Gamma(t) \quad (t > t_*) \\
& g_{t_2}(l) \leqslant g_{t_1}(l) \quad (t_2 \geqslant t_1 \geqslant t_*)
\end{aligned}
\tag{2.4}
$$

Any function $g_t(l, h_*)_v$ of the form

$$
g_t(l, h_*)_v = g_t(l) + l \cdot \int_{t_*}^{t} v[\tau]\, d\tau
$$

where $g_t(l)$ satisfies conditions (2.4) and $v[\tau]$ is some measurable function that satisfies condition $v[\tau] \in Q(\tau)$ $(\tau \geqslant t_*)$, is called admissible function for position $\{t_*, h_*\}$.

We identify the strategy U of the first player with the mapping

$$
U:\ \{t, h\} \to u[\cdot]
$$

which associates a certain function $u[\tau] (\tau \geqslant t)$ that is measurable in any segment and satisfies condition $u[\tau] \in P(\tau)$, to any position $\{t, h\}$. Let $\Delta = \{\tau_i:\ t_0 = \tau_0 < \tau_1 < \tau_2 < \ldots < \tau_m < \ldots\}$ be some subdivision of semiaxis $[t_0, \infty)$. In conformity with [3] the motion from position $\{t_0, h_0\}$ induced by strategy U with subdivision Δ is defined by function $h_t(l) = h_t(l; t_0, h_0, U, \Delta)$ which is determined by the following recurrent relationships:

$$
h_t(l) = g_t(l; h_{\tau_i}(\cdot))_v - l \cdot \int_{\tau_i}^{t} u_i[\tau]\, d\tau, \quad \tau_i < t \leqslant \tau_{i+1}
$$

where $h_{t_0}(l) = h_0(l)$, $u_i[\cdot] = U(\tau_i, h_{\tau_i}(\cdot))$, and $g_t(l; h_{\tau_i}(\cdot))_v$ is an arbitrary function admissible for position $\{\tau_i, h_{\tau_i}\} \in R^1 \times H$. The problem of convergence can now be formulated as follows.

Problem 1. Devise for position $\{t_0, h_0\}$ a strategy U_e which for any number $\varepsilon > 0$ determines a number $\delta = \delta(\varepsilon)$ such that the inclusion

$$
h_\vartheta(\cdot) \in L^{(\varepsilon)} \tag{2.5}
$$

would hold for any motion $h_t(l) = h_t(l; t_0, h_0, U_e, \Delta)$, provided the diameter of subdivision Δ does not exceed the number δ.

It was noted in [2-4] that condition (2.5) of translating all elements $h_\vartheta(\cdot)$ to the fairly small ε-neighborhood of set L in the metric of space H simultaneously ensures the translation of all information sets $G[\vartheta]$ to some fairly small, Euclidean neighborhood of set M. Problem 1, may be, therefore considered to be a formalization of the input control problem.

3. We present the conditions of solvability of Problem 1 in accordance with [2-4] and, also, describe the method used there for constructing the resolving strategy.

Let $W(t) \subset H$ $(t_0 \leqslant t \leqslant \vartheta)$ be a system of nonempty sets. We determine

$$\rho(t, h) = \inf \{\| h - y \|_H : y \in W(t)\} \tag{3.1}$$

Let $\{f_k\}$ be some minimizing sequence of elements for $\rho(t, h)$. We construct a sequence of elements $\eta_k \in H$ as follows:

$$\eta_k = \begin{cases} f_k(l) - h(l), & \text{if } f_k(l) < h(l) \\ 0, & \text{if } f_k(l) \geqslant h(l) \end{cases} \tag{3.2}$$

Let $\Omega(t, h)$ be the totality of all possible weak limits of all possible sequences η_k of form (3.2). We denote by $S(t, h)$ the following set of vectors:

$$S(t, h) = \left\{ s \in R^n : s = \int_\Theta l \cdot \eta(l)\, d\{l\},\ \eta(\cdot) \in \Omega(t, h) \right\}$$

and by $s_e(t, h)$ an arbitrary function that associates vector $s = s_e(t, h) \in S(t, h)$ to any position $\{t, h\}$, and then determine set $P_\tau^{(e)} = P_\tau^{(e)}(t, h)$ as the totality of vectors $u^* \in P(\tau)$ which satisfy the condition

$$u^* \cdot s_e(t, h) = \max_{u \in P(\tau)} u \cdot s_e(t, h)$$

Since $P(\tau)$ is measurable, sets $P_\tau^{(e)}$ are measurable in τ, when t and h are fixed (see, e.g., [6]).

The strategy $U^{(e)}$ extremal to the system of sets $W(t)$ $(t_0 \leqslant t \leqslant \vartheta)$ is determined by the mapping

$$U^{(e)} : \{t, h\} \to u[\cdot]$$

which associates to any position $\{t, h\}$ an arbitrary measurable function $u[\tau]$ $(\tau \geqslant t)$ that satisfies condition $u[\tau] \in P_\tau^{(e)} = P_\tau^{(e)}(t, h)$ $(\tau \geqslant t)$.

In conformity with [2-4] we call the system of sets $W(t) \subset H$ $(t_0 \leqslant t \leqslant \vartheta)$ u-stable, if for any $t_*, t^* \in [t_0, \vartheta]$, $t^* > t_*$, $h_* \in W(t_*)$, and admissible function $g_t(l; h_*)_v$, there exists a measurable function $u[t]$ $(t_* \leqslant t \leqslant t^*)$ with values in $P(t)$ such that

$$\left\{ g_{t^*}(l;\ h_*)_v - l \cdot \int_{t_*}^{t^*} u[\tau]\, d\tau \right\} \in W(t^*)$$

The function of extremal strategy is determined by the following lemma.

Lemma 1. If $W(t)$ $(t_0 \leqslant t \leqslant \vartheta)$ is a u-stable system of sets and $W(\vartheta) \subset L$, the strategy extremal for the indicated system of sets resolves Problem 1 for any position $\{t_0, h_0\}$, $h_0 \in W(t_0)$.

Thus to solve the problem of convergence for position $\{t_0, h_0\}$ it is sufficient to construct some u-stable system of sets that contains that position and at instant ϑ terminates in set L.

4. The program absorption construction described in [2-4], although similar to that in [7, 8], is modified to suit the particular characteristics of the problem of control with incomplete information. In the regular case the program absorption sets form a u-stable system of sets that terminate at instant ϑ in L, hence in conformity with Lemma 1 they can be used for solving Problem 1.

Another method is proposed for obtaining the maximal u-stable system of sets. It is similar to the method of retrograde constructions in [3, 5], but modified to suit the particular characteristics of the problem considered here.

As in [5], we determine for any arbitrary A and $B \subset H$ the operation of geometric difference

$$A \overset{*}{-} B = \{h \in H : h + B \subset A\} \tag{4.1}$$

As in the finite-dimensional case the relationships

$$\begin{aligned}(A \overset{*}{-} B) \overset{*}{-} C &= A \overset{*}{-} (B + C)\\ (A + B) \overset{*}{-} C &\supset (A \overset{*}{-} C) + B\end{aligned} \tag{4.2}$$

are valid.

Let G be some set in R^n. Let us determine the set

$$\Lambda(G) = \{h \in H : h(l) = l\cdot x,\ x \in G\}$$

The relationship

$$\Lambda(G_1 + G_2) = \Lambda(G_1) + \Lambda(G_2) \tag{4.3}$$

is evidently valid for arbitrary sets G_1 and $G_2 \subset R^n$ and the compactness or convexity of set G implies compactness or convexity of set $\Lambda(G) \subset H$, respectively.

We introduce mapping $\varphi_t : 2^H \to 2^H$ which associates to any set $A \subset H$ the set $\varphi_t(A)$ of elements $h \in H$ which have the following properties: any element $g \in H$, $g(l) \leqslant h(l)$, and $g \in \Gamma(t)$ satisfy the inclusion $g \in A$. Let $\Delta = \{\tau_i : t_0 = t_0 < \tau_1 < \ldots < \tau_m = \vartheta)$ be some subdivision of segment $[t_0, \vartheta]$. We determine sets $W_i \subset H$ $(i = 1, 2, \ldots, m)$ by the following recurrent formula:

$$W_i = \varphi_{\tau_{m-i+1}}\Big\{\Big(W_{i-1} + \Lambda\Big(\int\limits_{\tau_{m-i}}^{\tau_{m-i+1}} P(\tau)\, d\tau\Big)\Big) \overset{*}{-} \Lambda\Big(\int\limits_{\tau_{m-i}}^{\tau_{m-i+1}} Q(\tau)\, d\tau\Big)\Big\} \tag{4.4}$$

where it is assumed that $W_0 = L$. Note that owing to the completeness of sets $P(\tau)$ and $Q(\tau)$, and to conditions (1.3), the multiple-valued integrals of $P(\tau)$ and $Q(\tau)$ in formulas (4.4) are convex nonempty compacta in R^n [9].

Each of sets W_i in formulas (4.4) represents the cut by hyperplane $t = \tau_{m-i}$ of the program absorption of the proceding set W_{i-1} at instant τ_{m-i+1}.

The properties of sets W_i $(i = 1, 2, \ldots, m)$ approximate those of the u-stability, namely that for any $h_* \in W_i$ and function $g_t(l; h_*)_v$ $(\tau_{m-i} \leqslant t \leqslant \tau_{m-i+1})$ admissible for position $\{\tau_{m-i}, h_*\}$ there exists a measurable function $u[t]$ $(\tau_{m-i} \leqslant t \leqslant \tau_{m-i+1})$ with values in $P(t)$ such that

$$\Big\{g_{\tau_{m-i+1}}(l; h_*)_v - l\cdot \int\limits_{\tau_{m-i}}^{\tau_{m-i+1}} u[\tau]\, d\tau\Big\} \in W_{i+1}$$

Below we denote set W_m by W_Δ and define the set

$$W^c(\vartheta, t_0) = \bigcap_{\Delta} W_\Delta$$

where cross sections are taken at all possible subdivisions Δ of segment $[t_0, \vartheta]$. It can be shown that the formula

$$W^c(\vartheta, t_0) = \bigcap_{i=1}^{\infty} W_{\Delta_i} \tag{4.5}$$

is valid for any sequence of subdivisions $\{\Delta_i\}$ such that $\Delta_{i+1} \subset \Delta_i$ and $\Delta_i \neq \Delta_{i+1}$.

Let us consider the system of sets $W^c(t) = W^c(\vartheta, t)$ $(t_0 \leqslant t \leqslant \vartheta)$. The definition of $W^c(t)$ implies that $W^c(\vartheta) = L$ and, consequently, that the obtained system of sets terminates at instant ϑ in L. Bearing in mind the use of the derived system for solving Problem 1, we shall check the conditions of the u-stability of that system. Let $\Delta = \{\tau_i : t_1 = \tau_1 < \tau_2 < \ldots < \tau_m = t_2\}$ be some subdivision of segment $[t_1, t_2]$, L_* be an arbitrary set in H, and $W_\Delta{}^*$ a set constructed over the subdivision Δ by the recurrent formula (4.6), where we have to set $\vartheta = t_2$, $t_0 = t_1$, and $L = L_*$. Using the mapping of φ_t and formulas (4.2) and (4.3) we can prove by induction the validity of inclusion

$$W_\Delta{}^* \subset \varphi_{t_2}\left\{\left(L_* + \Lambda\left(\int_{t_1}^{t_2} P(\tau)\, d\tau\right)\right) \overset{*}{-} \Lambda\left(\int_{t_1}^{t_2} Q(\tau)\, d\tau\right)\right\} \tag{4.6}$$

To prove the u-stability properties of sets $W^c(t)$ $(t_0 \leqslant t \leqslant \vartheta)$ on the basis of formula (4.6) we use the following easily verified relationship:

$$\varphi_t\left(\bigcap_{i=1}^{\infty} A_i\right) = \bigcap_{i=1}^{\infty} \varphi_t(A_i) \tag{4.7}$$

$$\bigcap_{i=1}^{\infty} (A_i \overset{*}{-} B) = \left(\bigcap_{i=1}^{\infty} A_i\right) \overset{*}{-} B$$

$$\bigcap_{i=1}^{\infty} (A_i + \Lambda(G)) = \left(\bigcap_{i=1}^{\infty} A_i\right) + \Lambda(G)$$

$$(A_1 \supset A_2 \supset A_3 \supset \ldots \supset A_i \supset \ldots)$$

where in the first two formulas A_i and $B \subset H$ are assumed to be arbitrary and in the last one set G is assumed compact and sets A_i are closed and satisfy the conditions appearing in parentheses.

An important property of sets $W^c(t)$ $(t_0 \leqslant t \leqslant \vartheta)$ defined by formula

$$W^c(t_*) \subset \varphi_{t^*}\left\{\left(W^c(t^*) + \Lambda\left(\int_{t_*}^{t^*} P(\tau)\, d\tau\right)\right) \overset{*}{-} \Lambda\left(\int_{t_*}^{t^*} Q(\tau)\, d\tau\right)\right\} \tag{4.8}$$

$$(\forall t_*, t^* \in [t_0, \vartheta], \quad t^* > t_*)$$

is implied by formulas (4.5) and (4.6) with (4.7) taken into account. Formula (4.8) with allowance for (4.1) exactly defines the required u-stability property of the system of sets $W^c(t)$. Note that the definition of the recurent procedure (4.4) implies in itself the maximum of the system of sets $W^c(t)$ $(t_0 \leqslant t \leqslant \vartheta)$ on segment $[t_0, \vartheta]$.

The described here useful, and in principle general, method of constructing stable bridges proves to be rather difficult in general practical applications. It is, however, possible to give some specific examples of game problems of control in which the construction of such bridges reduces to fairly elementary operations. Because of this it is important to develop other, possibly less universal but easier to apply algorithms for constructing suitable stable sets for various, as far as possible general, particular cases in which the described here recurrent procedure is a reasonably effective tool for solving certain game problems.

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